**SRC Technical Note**
**1997 - 022**

**November 24, 1997**

# Short-Length Menger Theorems

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**Abstract**

We give short and simple proofs of the following two theorems by Galil and Yu [3]. Let $s$ and $t$ be two vertices in an $n$-node graph $G$.
(1) There exist $k$ edge-disjoint $s$-$t$ paths of total length $O(n\sqrt{k})$.
(2) If we additionally assume that the minimum degree of $G$ is at least $k$, then there exist $k$ edge-disjoint $s$-$t$ paths, each of length $O(n/k)$.

Let $G = (V, E)$ be an undirected $n$-node graph, with no parallel edges, and let $s$ and $t$ be two vertices of $G$ such that there exist $k$ edge-disjoint $s$-$t$ paths. Our goal is to give short proofs of the following two theorems of Galil and Yu [3].

**Theorem 1** *There exist $k$ edge-disjoint $s$-$t$ paths of total length $O(n\sqrt{k})$.*

**Theorem 2** *If we additionally assume that the minimum degree of $G$ is at least $k$, then there exist $k$ edge-disjoint $s$-$t$ paths, each of length $O(n/k)$.*

We view $G$ as a directed graph by replacing each undirected edge by two oppositely oriented directed edges. Our proof of the first theorem is based on a maximum flow algorithm of Even and Tarjan [2]; for our purposes, we need only consider its global structure. The algorithm of [2] runs in phases numbered $1, 2, \ldots$. In phase $d$, a residual graph is maintained as a layered directed graph: the endpoints of each edge lie either in the same layer or in adjacent layers, and the distance from $s$ to $t$ is equal to $d$. The algorithm finds augmenting $s$-$t$ paths of length $d$ in this layered graph until there exist no more such paths of length at most $d$; the phase then ends.

**Proof of Theorem 1.** We analyze the behavior of the Even-Tarjan algorithm for producing a flow of value $k$ in $G$. We prove an upper bound on the total length of all augmenting paths found; this also upper bounds the total length of the flow paths. We set $\ell = 2nk^{-1/2}$. We say that an augmenting path is of type 0 if its length is at most $\ell$, and of type $i$ ($i \geq 1$) if its length is between $2^{i-1} \cdot \ell$ and $2^i \cdot \ell$. The total length of all type 0 paths is at most $k\ell = 2n\sqrt{k}$. To bound the total length of all type $i$ paths, for $i \geq 1$, note that at the start of phase $2^{i-1} \cdot \ell$ of the Even-Tarjan algorithm, there is some pair of adjacent layers in the residual graph whose union contains at most $n/(2^{i-2} \cdot \ell)$ vertices. Between this pair of layers there can be at most $n^2/(2^{2i-2} \cdot \ell^2)$ edges, and hence at most this many augmenting paths can be produced from phase $2^{i-1} \cdot \ell$ onward. Thus the total length of all type $i$ paths is at most $(n^2 2^{-2i+2}\ell^{-2}) \cdot (2^i \ell) = 4n^2\ell^{-1}2^{-i} = 2n\sqrt{k}2^{-i}$, and so the total length of *all* augmenting paths is at most $2n\sqrt{k} + 2n\sqrt{k}\sum_{i\geq 1} 2^{-i} = 4n\sqrt{k}$. ■

For the proof of the second theorem, we consider the problem of finding a set of $k$ edge-disjoint $s$-$t$ paths in $G$ whose total length is minimum. Let $P_1, \ldots, P_k$

be such a set of paths; say that an edge $e$ is a *flow edge* if it is contained in some $P_i$, and a *non-flow edge* otherwise. Observe that if $(u, v)$ is a flow edge, then by the optimality of $P_1, \ldots, P_k$, $(v, u)$ is not a flow edge. Finding $k$ edge-disjoint paths of minimum total length is a minimum-cost flow problem; if we take its linear programming dual, we obtain dual variables $y_v$, one for each $v \in V$, which are integers with the following properties. (See e.g. [1].)

(1) If $(u, v)$ is a flow edge, then $y_v - y_u \geq 1$.

(2) If neither $(u, v)$ nor $(v, u)$ is a flow edge, then $y_u$ and $y_v$ differ by at most 1.

(3) We may assume without loss of generality that $y_s = 0$, $y_t \geq 0$, and for every $j$ in the interval $[0, y_t]$, there exists a node $v$ with $y_v = j$. We define $X_i = \{v : y_v = i\}$.

**Proof of Theorem 2.** We claim that for each $i \in [0, y_t - 3]$, $\left|\cup_{j=i}^{i+4} X_j\right| \geq \frac{1}{5}k$; the theorem will follow directly from this. By (3), there exist vertices $u \in X_{i+1}$, $v \in X_{i+2}$, and $w \in X_{i+3}$. Now, suppose that at most $\frac{1}{5}k$ flow paths pass through at least two of them. At least $3k - 3$ edges are incident to $\{u, v, w\}$; at most $4 \cdot \frac{1}{5}k + 2 \cdot \frac{4}{5}k = \frac{12}{5}k$ of these are flow edges. Hence there are at least $\frac{3}{5}k - 3$ non-flow edges incident to $\{u, v, w\}$; at least $\frac{1}{5}k - 1$ of these are incident to a single one of these vertices. By (2), the endpoints of these non-flow edges lie in $\cup_{j=i}^{i+4} X_j$, and hence $\left|\cup_{j=i}^{i+4} X_j\right| \geq \frac{1}{5}k$.

Otherwise, at least $\frac{1}{5}k$ flow paths pass through at least two of $u, v, w$. Now, by (1), at most one flow path can pass through both $u$ and $v$ or both $v$ and $w$; and at most one can pass directly from $u$ to $w$. Thus (again by (1)), at least $\frac{1}{5}k - 3$ of these paths must pass from $u$ to $w$ via distinct vertices in $X_{i+2}$. Hence $|X_{i+2}| \geq \frac{1}{5}k - 3$, and so again $\left|\cup_{j=i}^{i+4} X_j\right| \geq \frac{1}{5}k$. ■

# References


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[2] S. Even, R. Tarjan, "Network flow and testing graph connectivity," *SIAM J. Computing*, 4(1975), pp. 507–518.

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