Shape Sensitive Geometric Permutations*

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Abstract

We prove that a set of n unit balls in $\mathcal{R}^d$ admits at most *four* distinct geometric permutations, or line transversals, thus settling a long-standing conjecture in combinatorial geometry. The constant bound significantly improves upon the $\Theta(n^{d-1})$ bound for the balls of arbitrary radii. Intrigued by this large gap between the two bounds, we also investigate how the number of geometric permutations varies as a function of shape, size, and spacing of objects. Our results include a tight bound of 2^{d-1} on the geometric permutations of n disjoint rectangular boxes in $\mathcal{R}^d$, and a constant bound on the geometric permutations for disks in the plane when the ratio between the largest to smallest disks is bounded. An important consequence of the former theorem is that if the smallest bounding boxes containing a set of geometric objects in $\mathcal{R}^d$ are pairwise disjoint, then those objects admit only 2^{d-1} permutations, which is a significant improvement on the $O(n^{2d-2})$ bound known for general convex objects.

1 Introduction

A line ℓ is called a *line transversal* for a set $\mathcal{S}$ of bodies in $\mathcal{R}^d$ if it intersects every member of $\mathcal{S}$. If the objects in $\mathcal{S}$ are pairwise disjoint convex bodies, then a line transversal of $\mathcal{S}$ induces two linear orderings, which are the orders in which ℓ meets the members of $\mathcal{S}$. The two orders induced by a line transversal are essentially the same (one is reverse of the other), and so they are together called a single *geometric permutation*. Let $g_d(n)$ denote the maximum number of geometric permutations, where the maximum is taken over all such families $\mathcal{S}$ of size n in $\mathcal{R}^d$. Then, the following asymptotic bounds on $g_d(n)$ have been known for some time:

1. $g_2(n) = 2n-2$ [4].
2. $g_d(n) = \Omega(n^{d-1})$ [7].
3. $g_d(n) = O(n^{2d-2})$ [11].

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The planar case of geometric permutations was first considered by Katchalski, Lewis, and Zaks [8], who constructed a family of $n \geq 4$ pairwise disjoint convex sets in $\mathcal{R}^2$ that has $2n-2$ permutations. (Figure 1 reproduces their construction—it consists of two disks and $n-2$ segments.) Subsequently, Edelsbrunner and Sharir [4] were able to show that $2n-2$ is also the maximum possible number of geometric permutations in the plane.

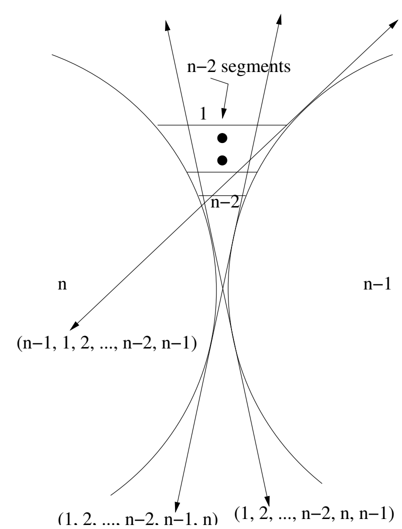

Figure 1: An example of n convex objects admitting $2n-2$ geometric permutations. (borrowed from [8]).

Katchalski, Lewis, and Liu [7] later generalized their lower-bound construction to d dimensions and showed that n disjoint convex bodies in $\mathcal{R}^d$ can realize $\Omega(n^{d-1})$ geometric permutations. The best upper bound known for general convex bodies is $O(n^{2d-2})$ by Wenger [11]. Thus, there remains a substantial gap between the known upper and lower bounds on geometric permutations.

This gap motivated Smorodinsky, Mitchell, and Sharir [9] to consider a natural, but specialized, family of convex bodies: balls. It turns out that even for families of balls $\Omega(n^{d-1})$ geometric permutations are realizable. Smorodinsky et al. [9] were able to prove a matching asymptotic upper bound, thus showing that the maximum number of geometric permutations for collections of n disjoint balls in $\mathcal{R}^d$ is $\Theta(n^{d-1})$.

An interesting special case of the problem is when all balls in $\mathcal{S}$ are congruent (say, unit balls). Smorodinsky et al. [9] show that the maximum number of geometric permutations for n unit discs in the plane is *only 2*, for sufficiently large n. (An independent proof of this fact is also given by Katchalski and Asinowski [2].) It has been conjectured that unit balls in any dimension admit only a constant number of permutations, but this conjecture has been open for dimensions $d \geq 3$.

While in combinatorial geometry line transversals and geometric permutations have been studied primarily for their mathematical appeal, they also find practical applications in computing visibility information in 3-dimensional computer games and architectural walkthroughs. These applications decompose the space into boxes, and then try to compute visibility between various boxes. The openings through which light can pass, such as windows, doors and the tops of stairwells in a building model, are modeled by axis-aligned rectangular *portals* on the faces of these boxes; all other objects are considered opaque. The visibility among boxes is computed from the visibility between these portals. See Teller and Sequin [10] for details. Similar ideas are used in various video games as well.

The portal visibility approach motivated research on the problem of determining whether a given set of objects can be stabbed by a line; that is, does the set have a line transversal. (The line transversal problem has also been studied as a generalization of the classical Helly Theorem of combinatorial geometry [3].) Amenta [1] and Hohmeyer and Teller [5] have given efficient algorithms for computing line transversals.

1.1 Our Contributions. In this paper, we are concerned with the combinatorial bounds for the number of distinct line transversals, or geometric permutations, for a set of disjoint convex bodies. Our first main result settles a long-standing open conjecture by proving that the maximum number of geometric permutations for a family of n pairwise disjoint unit radius balls in $\mathcal{R}^d$ is four, for n a sufficiently large constant that depends on the dimension d. The number of permutations, however, is *independent of d*. Thus, the number of geometric permutations drops precipitously from $\Theta(n^{d-1})$ to a small constant if all the balls are congruent.

On closer examination, we noticed that the lower bound construction for $\Omega(n^{d-1})$ permutations requires *exponential* ratio between the radii of largest to smallest balls. Since in practice the radii cannot differ so dramatically, we investigated the permutation problem when the ratio between the largest and smallest balls is bounded. It is believed that if the ratio between the radii of the largest and the smallest balls is bounded, then the number of permutations is also bounded. We prove this conjecture for the planar case.

Next, we consider the effects of shape and separation on the number of geometric permutations. We show that the maximum number of geometric permutations for a set of n pairwise disjoint rectangular boxes in d dimensions is 2^{d-1}. We also give a lower bound construction showing that this bound is tight.

A useful, and quite practical, corollary of the preceding box result is the following, which exploits pairwise separation among the objects. Let $\mathcal{S}$ be a set n pairwise disjoint convex bodies in d-space such that the smallest enclosing bounding boxes of these objects are also pairwise disjoint. Then, the number of geometric permutations for $\mathcal{S}$ is at most 2^{d-1}! In other words, even arbitrary convex bodies admit only a constant number of permutations *if* they are relatively well-separated. By contrast, without the separation assumption, the best upper bound known for permutations of convex objects is $O(n^{2d-2})$.

Our paper is organized as follows. In Section 2, we prove the upper bound on the number of geometric permutations for congruent balls. In Section 3, we considered the case when the ratio between the largest to the smallest ball is bounded. In Section 4, we prove the bound on permutations of rectangular boxes. In Section 5, we extend the proof of boxes to families with small size separating sets. In Section 6, we briefly consider the geometric permutation problem when the objects overlap, and finally in Section 7 we conclude. Due to lack of space, we omit several proofs from this extended abstract; all the details can be found in the technical report [12].

2 Geometric Permutations of Congruent Balls

In this section, we prove that a family of n pairwise disjoint unit radius balls in $\mathcal{R}^d$ can admit at most four distinct geometric permutations (independent of the dimension), when n is a sufficiently large constant that depends only on d. The proof is somewhat technical, and follows from a sequence of algebraic lemmas. The basic idea is to first show that the angle between any two line transversal is quite small; we then show that if two line transversals meet two balls in different orders, then these lines must pass very close to the center of gravity of the two balls; next, we establish distance bounds on the pairs of balls (called *switched pairs*) that appear in different orders in two different permutations; the proof is then completed by *lower bounding* the distance between two different switched pairs.

LEMMA 2.1. *Let $\mathcal{S}$ be a set of n disjoint unit balls* *in $\mathcal{R}^d$ with at least one geometric permutation. Then, the distance between the centers of the farthest pair of balls is $\Omega(n)$, where the constant depends on d.*

Proof. Let o_1, o_2 denote the centers of the two balls that form the farthest pair in $\mathcal{S}$, let $D = \mathrm{dist}(o_1, o_2)$, and let ℓ be a line transversal of $\mathcal{S}$. The projection of $\mathcal{S}$ onto ℓ has length at most $2 + D$, and thus the set $\mathcal{S}$ is contained within a radius 2 cylinder of height $2 + D$. If V_d denotes the volume of a unit ball in $\mathcal{R}^d$, then the volume of this cylinder is $V_{d-1}(2 + D)2^{d-1}$. Since n unit balls are contained in this cylinder, and d is a constant we get $V_{d-1}(2+D)2^{d-1} \geq nV_d$, which implies $D \geq \frac{V_d}{2^{d-1}V_{d-1}}n - 2 = \Omega(n)$. This completes the proof. $\diamond$

LEMMA 2.2. *Let B_1, B_2 be two disjoint unit balls in $\mathcal{R}^d$ with centers o_1 and o_2, and let ℓ be a line that intersects them both. Let $D = \mathrm{dist}(o_1, o_2)$, and let θ denote the angle between ℓ and the line containing the two centers o_1, o_2. Then, we have $\sin\theta \leq \frac{2}{D}$, and θ achieves its maximum value when ℓ is a common tangent of B_1, B_2.*

Proof. We use the line containing o_1, o_2 as our first coordinate axis, and call it the z-axis. Thus θ is the angle between ℓ and z-axis. Let o_1 be the origin, and so $\overrightarrow{o_1o_2} = (D, 0, \cdots, 0)$. Let p_1, p_2 be two arbitrary points where ℓ intersects the balls B_1, B_2. Clearly, the lengths of o_1p_1 and o_2p_2 are both bounded by 1. We can write $\overrightarrow{p_1p_2} = \overrightarrow{o_1o_2} + \overrightarrow{o_2p_2} + \overrightarrow{p_1o_1} = (D+x_1, x_2, \cdots, x_d)$, where $x = (x_1, \cdots, x_d) = \overrightarrow{o_2p_2} + \overrightarrow{p_1o_1}$, and so $|x| \leq 2$. Now,

$$\begin{aligned}\cos\theta &= \frac{D(D + x_1)}{D\sqrt{(D + x_1)^2 + x_2^2 + \cdots + x_d^2}} \\ &= \frac{D + x_1}{\sqrt{(D + x_1)^2 + x_2^2 + \cdots + x_d^2}}\end{aligned}$$

Thus,

$$\begin{aligned}\sin^2\theta &= \frac{x_2^2 + \cdots + x_d^2}{(D + x_1)^2 + x_2^2 + \cdots + x_d^2} \\ &= \frac{y}{(D + x_1)^2 + y},\end{aligned}$$

where $y = x_2^2 + \cdots + x_d^2$. Denote $f(y) \equiv \frac{y}{c+y}$. Then $f(y)$ is increasing with $y \geq 0$ when c is a positive constant. Because $y \leq 4 - x_1^2$, we have $f(y) \leq f(4 - x_1^2)$. Therefore

$$\begin{aligned}\sin^2\theta &= \frac{y}{(D + x_1)^2 + y} \\ &= f(y) \\ &\leq f(4 - x_1^2) \\ &= \frac{4 - x_1^2}{D^2 + 4 + 2Dx_1} \\ &\leq \frac{4}{D^2}.\end{aligned}$$

The last inequality follows from the following observations:

$$\begin{aligned}&\frac{4 - x_1^2}{D^2 + 4 + 2Dx_1} \leq \frac{4}{D^2} \\ \Leftrightarrow\quad &(4 - x_1^2)D^2 \leq 4(D^2 + 4 + 2Dx_1) \\ \Leftrightarrow\quad &(Dx_1 + 4)^2 \geq 0.\end{aligned}$$

Thus, $\sin^2\theta \leq \frac{4}{D^2} \Rightarrow \sin\theta \leq \frac{2}{D}$. The equality holds when $x_1^2 + \cdots + x_d^2 = 4$, and $Dx_1 + 4 = 0$. It can be verified easily that in this case line ℓ is a common tangent of the two balls. $\diamond$

LEMMA 2.3. *Let $\mathcal{S}$ be a set of n pairwise disjoint unit balls in $\mathcal{R}^d$ with at least one geometric permutation, and assume that z-axis is the line connecting the centers of the farthest pair of balls in $\mathcal{S}$. Then the angle between any line transversal of $\mathcal{S}$ and the z-axis is bounded by $O(1/n)$; consequently, the angle between any two line transversals of $\mathcal{S}$ is also $O(1/n)$.*

Proof. Let o_1, o_2 denote the centers of the two balls that form the farthest pair, and let $D = \mathrm{dist}(o_1, o_2)$. By Lemma 2.1, we have that $D = \Omega(n)$. Given an arbitrary line transversal ℓ, if θ is the angle between ℓ and the z-axis, then by Lemma 2.2 we have $\sin\theta \leq \frac{2}{D}$. Thus,

$$\theta \leq \frac{\pi}{2}\sin\theta \leq \frac{\pi}{D} = O(1/n),$$

where the last inequality uses the fact that $D = \Omega(n)$. The angle between any two line transversals of $\mathcal{S}$ is at most twice the maximal angle between one line transversal and the z-axis, and so the lemma follows. $\diamond$

Our next lemma derives an algebraic condition for the intersection of a line and a ball in a parametric form that is useful for our proof.

LEMMA 2.4. *Let B be a unit ball in $\mathcal{R}^d$ with center $o = (o_1, o_2, \cdots, o_d)$. Let $\ell = \{\ell(t) \mid t \in \mathcal{R}\}$ be the parametric equation of a line with $\ell(t) = (t, at + b, c_3, \cdots, c_d)$. If ℓ has a nonempty intersection with B, then $\ell(t_0) \in B$, where $t_0 = \frac{o_1 + ao_2 - ab}{a^2+1}$, and*

$$\frac{(ao_1 + b - o_2)^2}{1 + a^2} + \sum_{3 \leq i \leq d}(c_i - o_i)^2 \leq 1 \ .$$

Proof. In order for line ℓ to intersect the unit ball B, it is necessary and sufficient that

$$\min_{t \in \mathcal{R}}\left[(t - o_1)^2 + (at + b - o_2)^2 + \sum_{3 \leq i \leq d}(c_i - o_i)^2\right] \leq 1.$$

We note that

$$\begin{aligned} f(t) &= (t-o_1)^2 + (at+b-o_2)^2 \\ &= (a^2+1)t^2 + 2t(ab-ao_2-o_1) + o_1^2 + \\ &\quad (b-o_2)^2 \\ &= (a^2+1)\left(t - \frac{o_1+ao_2-ab}{a^2+1}\right)^2 + \\ &\quad \frac{(ao_1+b-o_2)^2}{1+a^2}. \end{aligned}$$

Thus, the minimal value of $f(t)$ is $\frac{(ao_1+b-o_2)^2}{1+a^2}$, and it is reached when $t = (o_1+ao_2-ab)/(a^2+1)$. So, in order for ℓ to intersect B, it is necessary and sufficient that $\ell(t_0) \in B$, where $t_0 = (o_1+ao_2-ab)/(a^2+1)$ and

$$\frac{(ao_1+b-o_2)^2}{1+a^2} + \sum_{3\le i\le d}(c_i-o_i)^2 \le 1.$$

$\diamond$

The preceding technical lemma is used to prove the following important fact, which intuitively states the following: if two line transversals intersect two balls in different orders, then these two lines must pass *very close* to the center of gravity of the two balls—the center of gravity of the two balls is just the midpoint of the segment joining their centers. This technical fact is used crucially in the proofs of two key lemmas: Lemma 2.8 and Theorem 2.1.

LEMMA 2.5. *Let P and Q be two disjoint unit balls in $\mathcal{R}^d$. Assume that their mutual center of gravity has its first coordinate equal to zero, and that the first coordinate axis $\vec{z}$ intersects both P and Q, with P first. If another line $\ell = \{\ell(t) = (t, at+b, c_3, \cdots, c_d) \mid t \in \mathcal{R}\}$ intersects both P and Q, with Q first, then we must have $|b| \le a$, where $a > 0$.*

Proof. Suppose that the centers of P and Q have coordinates $(p_1,\ p_2,\ \cdots,\ p_d)$ and $(q_1,\ q_2,\ \cdots,\ q_d)$ respectively. From the given condition, $p_1 + q_1 = 0$. So that we rewrite $p_1 = -\epsilon, q_1 = \epsilon$. Because the first coordinate axis $\vec{z}$ intersects both P and Q, we have

$$\sum_{2\le i\le d} p_i^2 \le 1 \qquad \text{and} \qquad \sum_{2\le i\le d} q_i^2 \le 1. \tag{2.1}$$

And because it intersects P before Q, we have $-\epsilon < \epsilon$, or $\epsilon > 0$.

Because line ℓ intersects Q, from Lemma 2.4, thus $\ell(t_2) \in Q$, where

$$t_2 = \frac{\epsilon + aq_2 - ab}{a^2+1},$$

and furthermore

$$\frac{(a\epsilon+b-q_2)^2}{1+a^2} + \sum_{3\le i\le d}(c_i-q_i)^2 \le 1. \tag{2.2}$$

Similarly, because ℓ intersects P, we have $\ell(t_1) \in P$, where

$$t_1 = \frac{-\epsilon + ap_2 - ab}{a^2+1},$$

and

$$\frac{(-a\epsilon+b-p_2)^2}{1+a^2} + \sum_{3\le i\le d}(c_i-p_i)^2 \le 1. \tag{2.3}$$

Because ℓ intersect Q first, we have $t_2 < t_1$. By plugging in the expressions for t_1, t_2, we get

$$\frac{\epsilon + aq_2 - ab}{a^2+1} < \frac{-\epsilon + ap_2 - ab}{a^2+1}$$

$$\Rightarrow \quad 2\epsilon < a(p_2-q_2) \ \Rightarrow\ \epsilon < \frac{a(p_2-q_2)}{2}. \tag{2.4}$$

By adding (2.2) and (2.3) together, we get

$$\begin{aligned} &\frac{(a\epsilon+b-q_2)^2 + (-a\epsilon+b-p_2)^2}{1+a^2} \\ &+ \sum_{3\le i\le d}\left[(c_i-q_i)^2 + (c_i-p_i)^2\right] \ \le\ 2. \end{aligned} \tag{2.5}$$

Because

$$(c_i-q_i)^2 + (c_i-p_i)^2 \ge \frac{(p_i-q_i)^2}{2}, \quad 3 \le i \le d,$$

we have

$$\begin{aligned} &\frac{(a\epsilon+b-q_2)^2 + (-a\epsilon+b-p_2)^2}{1+a^2} \\ &+ \sum_{3\le i\le d}\frac{(p_i-q_i)^2}{2} \le 2 \\ \Leftrightarrow\ &\frac{(a\epsilon+b-q_2)^2 + (-a\epsilon+b-p_2)^2}{2(1+a^2)} \\ &\le 1 - \sum_{3\le i\le d}\frac{(p_i-q_i)^2}{4} \end{aligned} \tag{2.6}$$

Because P, Q are disjoint, we get

$$4\epsilon^2 + \sum_{2\le i\le d}(p_i-q_i)^2 \ge 4 \tag{2.7}$$

$$\Leftrightarrow \quad 1 - \sum_{3\le i\le d}\frac{(p_i-q_i)^2}{4} \le \epsilon^2 + \frac{(p_2-q_2)^2}{4}. \tag{2.8}$$

By combining (2.6) and (2.8) together, we get

$$\begin{aligned}
&\frac{(a\epsilon+b-q_2)^2+(-a\epsilon+b-p_2)^2}{2(1+a^2)} \le \epsilon^2+\frac{(p_2-q_2)^2}{4} \\
\Leftrightarrow\quad & b^2-b(p_2+q_2)+a^2\epsilon^2+\frac{p_2^2+q_2^2}{2}+a\epsilon(p_2-q_2) \\
&\le (a^2+1)\left(\epsilon^2+\frac{(p_2-q_2)^2}{4}\right) \\
\Leftrightarrow\quad & b^2-b(p_2+q_2)+\frac{(p_2+q_2)^2}{4} \\
&\le \epsilon^2-a\epsilon(p_2-q_2)+a^2\frac{(p_2-q_2)^2}{4} \\
\Leftrightarrow\quad & \left(b-\frac{p_2+q_2}{2}\right)^2 \le \left(\epsilon-\frac{a(p_2-q_2)}{2}\right)^2.
\end{aligned}$$

With the help of (2.4), the preceding inequality immediately implies that

$$\left|b-\frac{p_2+q_2}{2}\right| \le \frac{a(p_2-q_2)}{2}-\epsilon \tag{2.9}$$

From (2.1) and (2.7), we get

$$\begin{aligned}
&\sum_{2\le i\le d}(p_i+q_i)^2 \\
&= 2\left[\sum_{2\le i\le d}(p_i^2+q_i^2)\right]-\left[\sum_{2\le i\le d}(p_i-q_i)^2\right] \\
&\le 4-(4-4\epsilon^2)=4\epsilon^2
\end{aligned} \tag{2.10}$$

Eq. (2.10) implies that $|p_2+q_2|\le 2\epsilon$. By combining it with (2.9), we get

$$|b| \le \frac{|p_2+q_2|}{2}+\frac{a(p_2-q_2)}{2}-\epsilon \le \frac{a(p_2-q_2)}{2} \le a.$$

This completes the proof. $\diamond$

2.1 Switched Pairs. We say that a pair of balls B_1, B_2 forms a *switched pair* in $\mathcal{S}$ if there exist two line transversals ℓ and ℓ' that meet B_1, B_2 in different orders. Figure 2 shows an example of a switched pair (B_1, B_2). Switched pair are obviously central to geometric permutations, and our next few lemmas establish key properties that allow us to bound their number. Our first lemma gives an upper bound on the distance between two balls forming a switched pair. The distance between two balls P and Q is defined as their point-wise minimum distance: $\mathrm{dist}(P,Q) = \min_{p\in P, q\in Q}\mathrm{dist}(p,q)$.

Lemma 2.6. *The distance between a switched pair is $O(1/n^2)$.*

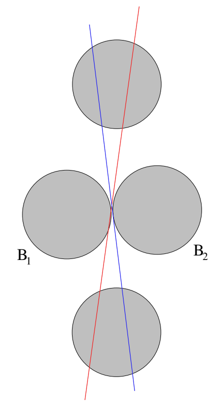

Figure 2: An example of a switched pair (B_1, B_2).

Lemma 2.7. *Let (B_1, B_2) be a switched pair in $\mathcal{S}$, and let o_1, o_2 be the centers of these balls. If ℓ is a line transversal of $\mathcal{S}$, then the line ℓ is nearly perpendicular to the line passing through o_1, o_2. More specifically, the angle between ℓ and $\overrightarrow{o_1o_2}$ is $\pi/2 - O(1/n)$.*

The following is one of the central facts used in our proof.

Lemma 2.8. *Let $\mathcal{S}$ be a set of n pairwise disjoint unit balls in $\mathcal{R}^d$, where n is a sufficiently large constant depending on d. Then, a ball of $\mathcal{S}$ can appear in at most one switched pair.*

Proof. We prove the unique pairing argument by contradiction. Suppose that $\mathcal{S}$ is a set of disjoint unit balls in $\mathcal{R}^d$ with at least one line transversal. Let $\vec{z}$-axis denote the line connecting the centers of two farthest balls of $\mathcal{S}$. Suppose that ball B forms switched pairs with at least two other balls. Specifically, assume that both (B, B_1) and (B, B_2) are switched pairs, where B, B_1, B_2 are distinct balls in $\mathcal{S}$, with centers o, o_1, o_2 respectively.

By Lemma 2.6, $\mathrm{dist}(B, B_1) = O(1/n^2)$ and $\mathrm{dist}(B, B_2) = O(1/n^2)$. Because n is sufficiently large, $\mathrm{dist}(o, o_1)$ and $\mathrm{dist}(o, o_2)$ are close to 2. By Lemma 2.7, $\overrightarrow{oo_1}$ and $\overrightarrow{oo_2}$ are almost perpendicular to the $\vec{z}$ axis. If π is the plane passing through o_1 and perpendicular to $\vec{z}$, then both o_2, o_3 are very close to π. Let g_1 be the middle point of oo_1, and g_2 be the middle point of oo_2. Then g_1g_2 is parallel to o_1o_2 and thus very close to π. Figure 3 illustrates this in the limiting case of $n = \infty$. In addition, $\mathrm{dist}(g_1, g_2) = \mathrm{dist}(o_1, o_2)/2 \ge 1$. By Lemma 2.5, any line transversal ℓ should pass close to g_1, the center of gravity of (B, B_1), as well as g_2, the center of gravity of (B, B_2). Because ℓ is very close to both g_1, g_2, it is almost parallel to g_1g_2, or equivalently, ℓ should be almost perpendicular to $\vec{z}$. This is a contradiction, as the angle between ℓ and $\vec{z}$ is bounded by $O(1/n)$, from

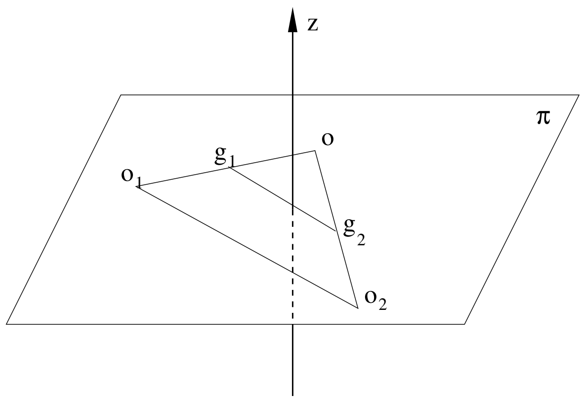

Figure 3: Illustration of the unique pairing property with $n = \infty$.

Lemma 2.3. ◇

The preceding lemma now easily implies the fact that if there are m switched pairs, then there can be at most 2^m permutations—each permutation is defined by one of the two orders of some switched pairs.

LEMMA 2.9. *Let $\mathcal{S}$ be a set of n disjoint convex objects in $\mathcal{R}^d$ where the unique pairing property holds for switched pairs. Then, two members forming a switched pair should always appear consecutively in all line transversals of $\mathcal{S}$. And if there are a total of m switched pairs, then $\mathcal{S}$ admits at most 2^m different geometric permutations.*

2.2 The Upper Bound.

LEMMA 2.10. *Consider two switched pairs: $P = (B_1, B_2)$ and $P' = (B_1', B_2')$. Then, there must exist two line transversals ℓ and ℓ' that meet both switched pairs in different orders. In other words, there exist line transversals ℓ and ℓ' such that ℓ meets B_1 before B_2 and B_1' before B_2', while ℓ' meets B_2 before B_1 and B_2' before B_1'.*

Proof. By definition, for the switched pair $P = (B_1, B_2)$, there exists two line transversals ℓ and ℓ' such that ℓ meets B_1 before B_2 and ℓ' meets B_2 before B_1. Now consider the order in which ℓ and ℓ' meet the pair $P' = (B_1', B_2')$. If they meet the balls of P' in different orders, then ℓ and ℓ' are our desired line transversals. Otherwise, without loss of generality, let us assume that ℓ and ℓ' both meet B_1' before B_2'.

Since $P' = (B_1', B_2')$ is also a switched pair, there exists a line transversal ℓ'', which meets B_2' before B_1'. Now consider the order in which ℓ'' meets the balls of $P = (B_1, B_2)$. If ℓ'' meets B_1 before B_2, then ℓ' and ℓ'' are the desired line transversals. Otherwise, ℓ'' meets B_2 before B_1, in which case ℓ and ℓ'' are the desired line transversals. This completes the proof. ◇

We need one more technical lemma.

LEMMA 2.11. *The distance between the centers of gravity of two switched pairs has a lower bound of $\sqrt{2} - \epsilon(n)$, where $\epsilon(n) > 0$ and $\lim_{n\to\infty} \epsilon(n) = 0$.*

THEOREM 2.1. *Let $\mathcal{S}$ be a set of n pairwise disjoint unit balls in $\mathcal{R}^d$ that admits at least one line transversal. With n sufficiently large, the number of switched pairs in $\mathcal{S}$ is at most 2.*

Proof. We transform the coordinate system so that z-axis is oriented along the line connecting the centers of the farthest pair of balls in $\mathcal{S}$. For each switched pair, we project its center of gravity onto the z-axis and call the projected coordinate the *height* of this switched pair. Thus, all the switched pairs of $\mathcal{S}$ can be linearly ordered using their heights. Let $P_1, P_2, \cdots, P_m$ denote the sorted order of these pairs.

By Lemma 2.10, there exist two line transversals ℓ, ℓ' such that the orders in which they meet the balls of P_1 and P_m are switched. Let us choose ℓ as the first coordinate axis, and consider the projection of ℓ' onto the plane normal to ℓ. Choose the direction of this projection as the second coordinate axis. We can write the equations for ℓ, ℓ' in the following form:

$$\begin{aligned} \ell &: \{\ell(t) = (t, 0, \cdots, 0) \mid t \in \mathcal{R}\}, \\ \ell' &: \{\ell'(t) = (t, at + b, c_3, \cdots, c_d) \mid t \in \mathcal{R}\}, \end{aligned}$$

where $a, b, c_3, \cdots, c_d$ are all constants.

Let $P_1 = (B_1, B_1')$ be the first switched pair, and $P_m = (B_m, B_m')$ be the last switched pair. Because ℓ and ℓ' have different transversal orders for both switched pairs, we can assume that ℓ meets B_1 before B_1', and that it meets B_m before B_m'. Then, ℓ' meets B_1' before B_1, and B_m' before B_m. In addition, denote the centers for B_1, B_1', B_m, B_m' as o_1, o_1', o_m, o_m' respectively. Let the middle point of $o_1 o_1'$ be g_1, and the middle point of $o_m o_m'$ be g_m. Let the first coordinates of g_1, g_m be x_{g_1} and x_{g_m} respectively. We can assume that $x_{g_1} < x_{g_m}$, that is, line ℓ meets P_1 first.

By Lemma 2.5, we have that $|ax_{g_1} + b| \le a$ and $|ax_{g_m} + b| \le a$. By combining these two inequalities together, we get $a|x_{g_m} - x_{g_1}| \le 2a$, or $x_{g_m} - x_{g_1} \le 2$. By Lemma 2.11, the minimal distance between the centers of gravity of two switched pairs is at least $\sqrt{2} - \epsilon(n)$. In the limit, when n approaches infinity, all these line transversals will converge to a line z' that is parallel to the z-axis, and the centers of gravity for all these switched pairs lie on z'. Since m is the total number of switched pairs, we must have

$$m - 1 \le \frac{2}{\sqrt{2}} + \epsilon(n) = \sqrt{2} + \epsilon(n).$$

Since m is an integer, we conclude that $m \leq 2$ when n is sufficiently large. Thus, in any dimension d, when n is sufficiently large, the set $\mathcal{S}$ can have at most 2 switched pairs. $\diamond$

We can now state our first main theorem.

THEOREM 2.2. *A set $\mathcal{S}$ of n pairwise disjoint unit balls in $\mathcal{R}^d$ admits at most 4 distinct geometric permutations, when n is sufficiently large.*

Proof. The theorem follows directly by combining Lemmas 2.9 and 2.1. $\diamond$

3 Geometric Permutations of Varying Radii Disks

The preceding theorem states that if all balls are congruent, then the number of geometric permutations is at most four. In sharp contrast, the number of permutations jumps to $\Theta(n^{d-1})$ if the balls have arbitrary radii [9]. In particular, even in the case of planar disks, the number of permutations is at most 2 if the disks are congruent, but can be $\Theta(n)$ otherwise. A closer examination of the problem reveals that constructions that achieve $\Omega(n)$ distinct permutations for disks require *exponentially varying* radii. Both from a practical as well as a mathematical perspective, it is an outstanding question as to how the number of permutations grows as a function of the varying radii. In particular, it has been conjectured that if the ratio between the largest and smallest ball is a constant, then the number of permutations should also remain a constant. In this section, we prove this conjecture for the planar case.

We begin by defining the *scale factor*, which measures the ratio between the largest and the smallest ball in the set. Given a set of balls $\mathcal{S} = \{B_1, \cdots, B_n\}$ in $\mathcal{R}^d$, the scale factor of $\mathcal{S}$ is defined as follows:

$$\sigma(\mathcal{S}) \;=\; \max_{i,j} \frac{\mathrm{vol}(B_i)}{\mathrm{vol}(B_j)} \;=\; \max_{i,j} \left(\frac{r_i}{r_j}\right)^d,$$

with r_i is the radius of ball B_i, for $1 \leq i \leq n$. Our main result is the following theorem.

THEOREM 3.1. *Let $\mathcal{S}$ be a set of n pairwise disjoint disks in $\mathcal{R}^2$, with scale factor $\sigma = \sigma(\mathcal{S})$. The set $\mathcal{S}$ admits at most two distinct geometric permutations if $n = \omega(\sigma^{5/4})$.*

Proof. Without loss of generality, we assume that $1 = r_1 \leq \cdots \leq r_n = a$; that is, $a^2 = \sigma$. Consider a line transversal ℓ, and consider a projection of all the disks on this line. If the projection has length L, then all the disks can be contained inside a cylinder of radius $2a$ and height L. Because all the balls are disjoint, we have $4aL \;\geq\; \pi n$, which implies that

$$L \;\geq\; \frac{\pi\, n}{4\, a}.$$

Let o_i be the center of B_i, r_i its radius, and suppose that B_1, B_2 are two balls in $\mathcal{S}$ with the longest center distance. Letting $D = \mathrm{dist}(o_1, o_2)$, it follows that $L \leq D + 2a$. Thus,

$$D \geq L - 2a \geq \frac{\pi\, n}{4\, a} - 2a = \Omega\left(\frac{n - a^2}{a}\right).$$

Let us assume that the z-axis is oriented along the line containing o_1, o_2. Let ϕ_0 denote the largest angle between the z-axis and any line transversal of $\mathcal{S}$. Then, it is easy to show that

$$\sin\phi_0 \leq \frac{r_1 + r_2}{D} \leq \frac{2\,a}{D},$$

which implies that

$$\phi_0 \;\leq\; \frac{2}{\pi}\sin\phi_0 \;=\; O\left(\frac{a}{D}\right) \;=\; O\left(\frac{a^2}{n - a^2}\right).$$

We next consider properties of a switched pair; in this case, the disks involved in the pair may have different radii. Suppose (P, Q) is a switched pair of disks, whose centers are o_p, o_q, and whose radii are r_p, r_q. Let $\delta = \mathrm{dist}(P, Q)$, and thus $\mathrm{dist}(o_p, o_q) = r_p + r_q + \delta$. As in Lemma 2.6, let θ_0 denote the maximal angle between a line transversal and the line connecting o_p and o_q. Then

$$\sin\theta_0 = \frac{r_p + r_q}{r_p + r_q + \delta},$$

which implies that

$$\delta = (r_p + r_q)\left(\frac{1}{\sin\theta_0} - 1\right) \leq 2a\left(\frac{1}{\sin\theta_0} - 1\right).$$

We can show that $2\theta_0 + 2\phi_0 \geq \pi$, and therefore $\sin\theta_0 \geq \cos\phi_0$. Thus,

$$\begin{aligned}\delta \;&\leq\; 2a\left(\frac{1}{\sin\theta_0} - 1\right) \;\leq\; 2a\left(\frac{1}{\cos\phi_0} - 1\right)\\ &=\; \frac{4a\,\sin^2\phi_0/2}{\cos\phi_0} \;=\; O(a\phi_0^2) \;=\; O\left(\frac{a^5}{(n - a^2)^2}\right).\end{aligned}$$

If $n = \omega(a^{2.5})$, then $\delta \to 0$. This means that the distance between two balls in a switched pair approaches zero when $n = \omega(a^{2.5})$, or equivalent, when $n = \omega(\sigma^{5/4})$. Similarly we can show that the line connecting o_p, o_q is almost parallel to z-axis. These properties imply that two balls forming a switched pair are always consecutive in any permutation.

Combining this with a fact proved by Katchalski et al. [6], which states that in a set of four pairwise disjoint convex bodies $\{a, b, c, d\}$, if (a, b, c, d) is a geometric permutation, then (b, a, d, c) cannot be a geometric permutation, we conclude that there is at most one switched pair in the planar case, when the number of disks is $n = \omega(\sigma^{5/4})$. Thus the number of geometric permutations is two. $\diamond$

4 Geometric Permutations for Rectangular Boxes

In this section, we consider geometric permutations for another simple but fundamental shape: *rectangular boxes.* Recall that a rectangular box in $\mathcal{R}^d$ is the product of d real intervals: $\Pi_{1\le i\le d}[a_i, b_i]$, where $-\infty < a_i \le b_i < \infty$ for $i = 1, 2, \ldots, d$.

LEMMA 4.1. *Let P and Q be two disjoint rectangular boxes in $\mathcal{R}^d$. There exists at least one dimension k, for $1 \le k \le d$, such that the projections of P and Q onto the kth linear space are disjoint.*

Proof. This follows from elementary geometry. See Figure 4 for illustration. Suppose that $P = \Pi_{1\le i\le d}[p_{i_1}, p_{i_2}]$ and $Q = \Pi_{1\le i\le d}[q_{i_1}, q_{i_2}]$. Thus, $[p_{i_1}, p_{i_2}]$ and $[q_{i_1}, q_{i_2}]$ are the projections of P and Q onto the ith linear space. $P \cap Q = \Pi_{1\le i\le d}\left([p_{i_1}, p_{i_2}] \cap [q_{i_1}, q_{i_2}]\right)$. Since $P \cap Q = \emptyset$, there is at least one k such that $[p_{k_1}, p_{k_2}] \cap [q_{k_1}, q_{k_2}] = \emptyset$. $\diamond$

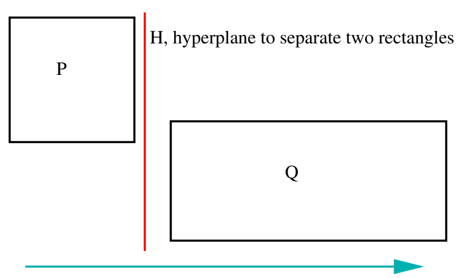

Figure 4: An example showing that two disjoint rectangular boxes can be separated by an axis parallel hyperplane.

For $\forall x \in \mathcal{R}$, denote

$$\mathrm{sign}(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{else.} \end{cases}$$

Furthermore, for $x = (x_1, \cdots, x_d) \in \mathcal{R}^d$, define $\mathrm{sign}(x) = (\mathrm{sign}(x_1), \cdots, \mathrm{sign}(x_d))$. The following lemma is easy to prove.

LEMMA 4.2. *Let $\mathcal{S}$ be a set of axis-parallel rectangular boxes in $\mathcal{R}^d$, and let ℓ, ℓ' are two line transversals of $\mathcal{S}$. Let $\vec{\ell}$ and $\vec{\ell'}$ be the vectors denoting the orientations of ℓ and ℓ'. If $sign(\vec{\ell}) = sign(\vec{\ell'})$, then ℓ and ℓ' induce the same linear ordering on $\mathcal{S}$.*

THEOREM 4.1. *Let $\mathcal{S}$ be a set of n pairwise disjoint axis-parallel rectangular boxes in $\mathcal{R}^d$. The maximum number of distinct geometric permutations admitted by $\mathcal{S}$ is 2^{d-1}.*

Proof. A line transversal ℓ, with fixed direction $\vec{\ell}$, induces a unique linear ordering on $\mathcal{S}$, and $\mathrm{sign}(\vec{\ell}) \in \{1, -1\}^d$. There are altogether 2^d elements in $\{1, -1\}^d$, and each of them corresponds to at most one linear ordering of $\mathcal{S}$ (by Lemma 4.2). Thus, the total number of linear orderings on $\mathcal{S}$ is at most 2^d. Note, however, that these linear orderings occur in pairs—each linear ordering also has a reverse ordering in the set (and for $n > 1$ the two orderings in each pair are different). Each such pair of linear ordering corresponds to one geometric permutations, and so the number of geometric permutations for a set of rectangular boxes in $\mathcal{R}^d$ is at most 2^{d-1}. $\diamond$

It turns out that the bound of 2^{d-1} is tight. The following theorem gives a matching construction, even using unit size cubes.

THEOREM 4.2. *There exists a set $\mathcal{S}$ of n axis-parallel unit cubes in $\mathcal{R}^d$ that admits at least 2^{d-1} distinct geometric permutations. The number of cubes needed is $n \ge 2d-2$ for $d > 2$, and $n > 2d-2$ for $d = 1$ or 2.*

Proof. The case where $d = 1$ is trivial, thus now we assume that $d \ge 2$. In the following we will construct $d-1$ pairs of close cubes (switched pairs), with the remaining $n - (2d-2)$ cubes lining up around the first coordinate axis. Specifically, we have $\mathcal{S} = A \cup B$ where $A = \{A_k, A'_k \mid 2 \le k \le d\}$ and $B = \{B_i \mid 1 \le i \le n - (2d-2)\}$. And the exact positions of A_k, A'_k, B_i are the following:

$$\begin{aligned} A_k &= [t_k, t_k+1] \times \Pi_{2\le i<k}[-1/2, 1/2] \times [-1-\epsilon, -\epsilon] \\ &\quad \times \Pi_{k<i\le d}[-1/2, 1/2], \\ A'_k &= [t_k, t_k+1] \times \Pi_{2\le i<k}[-1/2, 1/2] \times [\epsilon, 1+\epsilon] \\ &\quad \times \Pi_{k<i\le d}[-1/2, 1/2], \quad \text{for } 2 \le k \le d \\ B_i &= [t_{i+d}, t_{i+d}+1] \times \Pi_{1\le i\le d}[-1/2, 1/2], \\ &\quad \text{for } 1 \le i \le n-(2d-2). \end{aligned}$$

Here $t_i = 2i$ for all i's, and ϵ is a very small positive number which will be determined later. It is easy to see that these unit cubes are pairwise disjoint. Please refer to Figure 5.

Now for every $(1, \delta_2, \cdots, \delta_d)$ where $\delta_i \in \{-1, 1\}$, we want to construct one line transversal ℓ such that $\mathrm{sign}(\vec{\ell}) = (1, \delta_2, \cdots, \delta_d)$. Furthermore line ℓ induces one linear ordering on $\mathcal{S}$ with the following property: If $\delta_k = 1$, then A_k appears before A'_k, otherwise A'_k appears before A_k. Following is the construction:

$$\ell : \{\ell(t) = (t, \epsilon(a_2 t + b_2), \cdots, \epsilon(a_d t + b_d)) \mid t \in \mathcal{R}\}$$

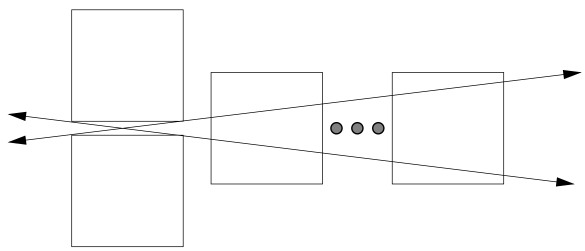

Figure 5: Lower bound construction for the case where $d = 2$.

Here $a_i = 2\delta_i$, $b_i = -\delta_i(1 + 2t_i)$, for $1 \le i < d$. Obviously $\vec{\ell}$ is proportional to $(1, 2\epsilon\, \delta_2, \cdots, 2\epsilon\, \delta_d)$ and $\mathrm{sign}(\vec{\ell}) = (1, \delta_2, \cdots, \delta_d)$. For $0 \le t \le t_n$ and $2 \le i \le d$, then the i-th coordinate of $\ell(t)$ is bounded by

$$|\epsilon(a_i t + b_i)| = |\epsilon(2t - 2t_i - 1)| \le \epsilon(2t_n) = 4n\epsilon.$$

If $0 < \epsilon < 1/(8n)$, then the above value is bounded by $1/2$. Here we choose $\epsilon = 1/(16n)$. For $1 \le i \le n - (2d - 2)$, if $t = t_{i+d}$, then $\ell(t) \in B_i$, and ℓ intersects B_i. Furthermore, if $t = t_k$ for $2 \le k \le d$, then the k-th coordinate of $\ell(t_k)$ is $\epsilon(a_k t_k + b_k) = -\delta_k\epsilon$. And for $t = t_k + 1$, the k-th coordinate of $\ell(t_k + 1)$ is $\epsilon(a_k(t_k + 1) + b_k) = \delta_k\epsilon$. By combining these computations with the above absolute value estimate for each coordinate, we get the following result: If $\delta_k = 1$, then $\ell(t_k) \in A_k$ and $\ell(t_k + 1) \in A'_k$; If $\delta_k = -1$, then $\ell(t_k) \in A'_k$ and $\ell(t_k + 1) \in A_k$. In summary, both A_k and A'_k intersect ℓ and the order for them to meet ℓ depends on δ_k.

There are totally 2^{d-1} possible elements in $\{(1, \delta_2, \cdots, \delta_d) \mid \delta_i \in \{-1, 1\}, 2 \le i \le d\}$. For each of them, we have a line transversal such that the order of A_k and A'_k depends only on δ_k. Thus all these linear orderings are different. For $d > 2$ case, there are $d - 1 \ge 2$ pairs of close cubes, and their orders are fixed no matter what δ_i is, thus there is no reverse orderings. For the case where $d = 2$, because $n > (2d - 2)$, thus B-family is not empty, and B-family cubes are always met after these A-family cubes, so that there is also no reverse orderings. In summary, these 2^{d-1} linear orderings result in different geometric permutations. So that there are at least 2^{d-1} geometric permutations. The proof is complete now. $\diamond$

5 Geometric Permutations using Separating Hyperplanes

In this section, we generalize the idea of using separating planes to obtain bounds on geometric permutations. The idea is not entirely new, and has been used by Smorodinsky, Mitchell, and Sharir [9] as well as Wenger [11], but in a different manner. Let us begin with a definition.

Let $\mathcal{S}$ be a family of pairwise disjoint objects in $\mathcal{R}^d$, and let $\mathcal{H}$ be a set of hyperplanes in $\mathcal{R}^d$ passing through the origin. We say that $\mathcal{H}$ is a *separation set* for $\mathcal{S}$ if, for each pair of objects $o_i, o_j \in \mathcal{S}$, there exists a hyperplane H, parallel to some hyperplane in $\mathcal{H}$, such that o_i, o_j are separated by H.

Wenger [11] shows that if $\mathcal{H}$ is a separation set for a collection of pairwise disjoint convex sets, then the number of geometric permutations admitted by $\mathcal{S}$ is $O(|\mathcal{H}|^{d-1})$. By contrast, we show that a separation set $\mathcal{H}$ bounds the number of geometric permutations by $2^{|\mathcal{H}|-1}$. Neither bound is necessarily dominated by the other. Our bound does not have the exponential dependence on d, though it does grow exponentially with $\mathcal{H}$. In many cases, as exemplified by Theorem 4.2, our result yields better bounds.

THEOREM 5.1. *Let $\mathcal{S}$ be a family of pairwise disjoint objects in $\mathcal{R}^d$, and let $\mathcal{H}$ be a separation set for $\mathcal{S}$. Then, the maximum number of geometric permutations admitted by $\mathcal{S}$ is $2^{|\mathcal{H}|-1}$.*

A very practical corollary of this result is formulated by the following theorem, which shows that if an arbitrary collection of convex objects are *reasonably well-separated*, then they admit significantly fewer permutations than the general bound of $O(n^{2d-2})$ would indicate.

THEOREM 5.2. *Let $\mathcal{S}$ be a set of n pairwise disjoint convex objects in $\mathcal{R}^d$ such that the smallest enclosing axis-parallel bounding boxes of these objects are also pairwise disjoint. Then, the set $\mathcal{S}$ can admit at most 2^{d-1} distinct geometric permutations.*

Another example where the separation sets lead to improved bounds is when the objects are homothets of a fixed convex polygon, or when all edges of the convex polygons are parallel to a small collection of directions.

THEOREM 5.3. *Let $\mathcal{S}$ be a set of disjoint convex polygons in $\mathcal{R}^2$, all of whose edges are parallel to a fixed set of m distinct directions. Then $\mathcal{S}$ admits at most 2^{m-1} geometric permutations.*

Unfortunately, this theorem does not extend to three or higher dimensions, as there are simple cases where two disjoint convex polyhedra cannot be separated by any hyperplane parallel to one of the faces of the polyhedra.

6 Geometric Permutations of Non-Disjoint Objects

Geometric permutations traditionally have been defined only for disjoint objects. Indeed, when objects overlap, there may not be a unique linear ordering induced by a line. In particular, if the line passes through a point on the common boundary of multiple objects, what is the natural order of these objects along the line? One can modify the definition of the linear order in many natural ways, and explore if they lead to a meaningful study of geometric permutations. Such a question is not just academic, because in several settings objects are not pairwise disjoint, and yet one is interested in defining a natural linear order. In this section, we explore this question to a limited extent.

One quite natural way to define line transversals of a set of objects, not necessarily pairwise disjoint, is to only consider lines that do not meet more then one object at any point. We will call a line transversal *degenerate* if it meets more than one object at any point, and *non-degenerate* otherwise. Each non-degenerate line transversal will induce at most two distinct permutations of all the objects, and we call them geometric permutations. Then, given a set of possibly intersecting objects, how many non-degenerate line transversals are possible?

THEOREM 6.1. *A set of n possibly overlapping rectangles in the plane admits at most $O(n^4)$ non-degenerate geometric permutations.*

Somewhat surprisingly, this bound is optimal—we can show a simple construction that achieves $\Omega(n^4)$ distinct permutations.

THEOREM 6.2. *There exists a set of n axis-parallel rectangles that admits $\Omega(n^4)$ non-degenerate geometric permutations.*

7 Concluding Remarks

We proved a number of results concerning geometric permutations for a set of disjoint convex bodies. Specifically, we resolved an open conjecture by showing that a set of n unit balls in $\mathcal{R}^d$ admits at most a *constant* number (four) of geometric permutations.

A general theme in our work was to explore the effects of shape, size, and separation on the complexity of permutations. While explicit constructions show that $\Omega(n^{d-1})$ distinct permutations are possible for n objects in $\mathcal{R}^d$, our research shows that, in many practical situations, one can prove significantly better bounds. For instance, if the bounding boxes of the convex bodies are pairwise disjoint, then the number of geometric permutations drops to just 2^{d-1}!

We also explored many extensions of the geometric permutation problems: objects with small separating sets, and overlapping objects. Many interesting open problems remain to be explored along these lines.

Finally, we want to acknowledge Jinhui Xu for pointing out a possible improvement in our constant bound for the permutations of unit balls.

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